Construction of five–dimensional orbifold gauge theories

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Outline

- Introduction to 5d $\mathrm{SU}(N)$ gauge theories
- The orbifold
- The boundary terms
- Lattice formulation
- Chiral fermions on the lattice
- Conclusion and Outlook

Introduction to 5d $\mathrm{SU}(N)$ gauge theories

- The Standard Model is a low–energy effective theory for energies $E \ll \Lambda$ where Λ is the scale of new physics
  - Hierarchy problem
    the Higgs mass receives radiative correction propotional to Λ^2
  - Triviality of the Higgs sector
    triviality bound: in ϕ^4 theory [Lüscher and Weisz, 1987]

$$\ln(\Lambda/m_{\mathrm{R}}) \quad \leq \quad A/g_{\mathrm{R}} + B\ln(g_{\mathrm{R}}) + \mathcal{O}(1)$$

    At fixed renormalized coupling g_{R} one cannot take $\Lambda \longrightarrow \infty$ *but* there is a scaling region where the amplitudes are universal up to effects E^2/Λ^2
- The Standard Model contains chiral fermions. Their non-perturbative definition is not obvious, since there is a no–go theorem (later in this talk). It can be bypassed starting from 5d and the lattice thus provides a gauge–invariant non–perturbative regulator
- We investigate 5d $\mathrm{SU}(N)$ gauge theories as possible extensions of the Standard Model

Introduction to 5d $\mathrm{SU}(N)$ gauge theories

Use of 5d gauge theories for extensions of the Standard Model: compact extra–dimension S^1 with radius $R \sim 1\,\mathrm{TeV}^{-1}$ [Antoniadis, 1990]

- Standard Model Higgs field is identified with the fifth dimensional component of the gauge field, elektroweak symmetry breaking by Hosotani mechanism [Hosotani, 1983]
- Dimensional reduction for energies $E \ll 1/R \ll \Lambda$, like in field theory at nonzero temperature
- Perturbative picture: Kaluza–Klein tower of states, 4d effective theory of zero modes ($p_5 = 0$), breaking of 5d gauge invariance as a Higgs mechanism [Karlhede and Tomaras, 1983], finite 1–loop Higgs mass and potential [Antoniadis and Benakli, 1994]

Open questions:

- Standard Model Higgs is not in the adjoint representation of the gauge group
- How can we take a 4d slice of a 5d theory?
- Is non–renormalizability of 5d gauge theories a problem?
- Non–perturbative?

Introduction to 5d $\mathrm{SU}(N)$ gauge theories

In d dimensions

$$S = \frac{1}{4}\int \mathrm{d}^d x \left[b\, F^A_{MN} F^A_{MN} + c\, D_L F^A_{MN} D_L F^A_{MN} \right]$$

$$F^A_{MN} = \partial_M A^A_N - \partial_N A^A_M + g_0 f^{ABC} A^B_M A^C_N$$

b and c are parameters. We introduce physical dimensions through rescaling with a momentum cutoff Λ

$$x = \Lambda x' , \quad A^A_M(x) = \zeta A'^A_M(x') , \quad g_0 = \xi g'_0$$

Requiring the standard coefficient 1/4 for the kinetic term fixes

$$\zeta = b^{-1/2} \Lambda^{(2-d)/2} \quad \text{and} \quad \xi = b^{1/2} \Lambda^{(d-4)/2}$$

and the action becomes

$$S = \frac{1}{4}\int \mathrm{d}^d x' \left[F'^A_{MN} F'^A_{MN} + \frac{c}{b}\frac{1}{\Lambda^2}\, D'_L F'^A_{MN} D'_L F'^A_{MN} \right]$$

$\Rightarrow$ higher dimensional terms suppressed by powers of E/Λ (E is a typical momentum)

Introduction to 5d $\mathrm{SU}(N)$ gauge theories

- $d = 5$ is trivial: $g_0'(\Lambda) = (b\Lambda)^{-1/2} g_0$ vanishes as $\Lambda \longrightarrow \infty$
  - 5d theory makes sense only at finite cutoff
  - Problem: smallness of the interactions, dismiss the theory?
  - An infinite flat fifth dimension is not physical
- Compactification of the fifth dimension, boundary conditions: dimensional reduction to 4d
- Does a range of the 5d cutoff exist, where 4d physics is universal up to corrections E/Λ? $\longrightarrow$ scaling region, effective low–energy theory
- The effective low–energy action is reminiscent of the Symanzik action, a local action in the continuum expanded in powers of the lattice spacing $a = 1/\Lambda$. It reproduces the lattice theory [Symanzik, 1981]

Introduction to 5d $\mathrm{SU}(N)$ gauge theories

Orbifold

Parametrize $S^1 : x_5 \in (-\pi R, \pi R]$, coordinates identified modulo $2\pi R$

- Projection $S^1 \longrightarrow S^1/\mathbb{Z}_2$ by identification of points under reflection $\mathcal{R} : x_5 \longrightarrow -x_5$
  - Fixpoints $x_5 = 0$ and $x_5 = \pi R$ define 4d boundaries
  - Projection embedded in a gauge theory can break the gauge symmetry at the fixpoints $\Rightarrow$ Higgs does not transform in the adjoint representation (S^1) but in some lower dimensional representation like in the Standard Model
- Recent phenomenological applications [Biggio, Feruglio, Masina and Perez–Victoria, 2004; Martinelli, Salvatori, Scrucca and Silvestrini, 2005]
- The renormalization of a theory can be changed by the presence of boundary counterterms [Symanzik, 1981; Lüscher, 1985]
- In this talk: derivation of the orbifold theory from a *gauge invariant* formulation $\Rightarrow$ simple classification of the boundary counterterms
- Non–perturbative definition of the orbifold gauge theory on the lattice is required to prove its phenomenological relevance

The orbifold

Gauge fields on S^1: two open charts

$$O^{(+)} = (-\epsilon, \pi R + \epsilon)\,,\; A_M^{(+)}\,,\; \Omega^{(+)}$$

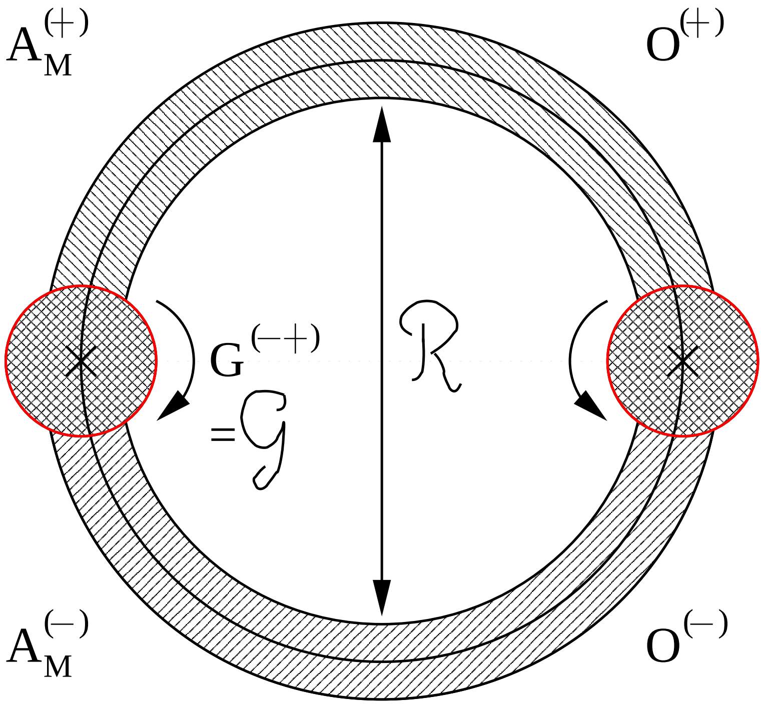
$$O^{(-)} = (-\pi R - \epsilon, \epsilon)\,,\; A_M^{(-)}\,,\; \Omega^{(-)}$$

$\Omega^{(\pm)}$ gauge transformations. On the overlaps of the two charts the gauge fields are related by gauge transformations or transition functions $G^{(-+)}$

$$A_M^{(-)} = G^{(-+)} A_M^{(+)} {G^{(-+)}}^{-1} + G^{(-+)} \partial_M {G^{(-+)}}^{-1}$$

Covariance

$$G^{(-+)} \longrightarrow \Omega^{(-)}\, G^{(-+)}\, {\Omega^{(+)}}^{-1}$$



The orbifold

Gauge covariant derivative of the transition functions

$$D_M G^{(-+)} \quad = \quad \partial_M G^{(-+)} + A_M^{(-)} G^{(-+)} - G^{(-+)} A_M^{(+)} \ \equiv \ 0!$$

Gauge fields on $S^1/\mathbb{Z}_2$

$\mathbb{Z}_2$ reflection $\mathcal{R}$ in the fifth direction, $z = (x_\mu, x_5)$, $\bar{z} = (x_\mu, -x_5)$

$$\mathcal{R}\, z \ = \ \bar{z}\,, \qquad \mathcal{R}\, A_M(z) \ = \ \alpha_M A_M(\bar{z})\,, \qquad \alpha_\mu = 1\,,\ \alpha_5 = -1$$

On the overlaps we *identify* $\mathcal{R}\, A_M^{(+)} = A_M^{(-)}$ and we define the gauge theory on

$$I_\epsilon \quad = \quad \{x_\mu, x_5 \in O^{(+)}\}$$

with gauge field $A_M \ \equiv \ A_M^{(+)}$ and a spurion field $\mathcal{G} \ \equiv \ G^{(-+)}$

$$\boxed{\mathcal{R}\, A_M \ = \ \mathcal{G}\, A_M\, \mathcal{G}^{-1} + \mathcal{G}\, \partial_M \mathcal{G}^{-1}\,, \qquad \mathcal{R}\, \mathcal{G} \ = \ \mathcal{G}^{-1}}$$

Constraining $\mathcal{R}\,\Omega = \Omega$, the fundamental domain of the $S^1/\mathbb{Z}_2$ gauge theory is the strip $I_0 = \{x_\mu, 0 \leq x_5 \leq \pi R\}$

The orbifold

At the fixpoints $x_5 = 0$ and $x_5 = \pi R$ of the reflection $\mathcal{R}$

$$\mathcal{G}^2 \quad = \quad 1$$

This suggest how to define the gauge breaking orbifold projection

- Start on I_ϵ and shrink the overlaps to the single fixpoints ($\epsilon \longrightarrow 0$)
- We have to specify what happens with the spurion field

$$\begin{aligned} \mathcal{G}(0) \;=\; \mathcal{G}(\pi R) \quad &= \quad g \\ \partial_5^p \mathcal{G}(0) \;=\; \partial_5^p \mathcal{G}(\pi R) \quad &= \quad 0\,, \quad p \in \mathbb{N}\,,\; p > 0 \end{aligned}$$

  for a *constant* matrix g obeying $g^2 = 1 \Rightarrow$ breaking of the gauge symmetry at $x_5 = 0$ and $x_5 = \pi R$

$$[\Omega, g] \quad = \quad 0$$

- We obtain boundary conditions on I_0 for *any* field at $x_5 = 0$ and $x_5 = \pi R$

$$\begin{aligned} \alpha_M\, A_M \quad &= \quad g\, A_M\, g \quad \text{Dirichlet boundary conditions} \\ -\alpha_M\, \partial_5 A_M \quad &= \quad g\, \partial_5 A_M\, g \quad \text{Neumann boundary conditions} \end{aligned}$$

The orbifold

Gauge symmetry breaking patterns by group conjugation with g (inner automorphism of the Lie algebra)

$$g = e^{-2\pi i V\cdot H}$$

$H = \{H_i\}\,,\ i = 1,\ldots,N-1$ are the Cartan generators, $V = \{V_i\}$ is a constant twist vector

$$G = SU(p+q) \quad \longrightarrow \quad \mathcal{H} = SU(p)\times SU(q)\times U(1)$$

- Hermitian $\mathrm{SU}(N)$ generators T^A, $A = 1,\ldots,N^2-1$ transform as

$$g\,T^A\,g \;=\; \eta^A\,T^A\,,\quad \eta^A = \pm 1$$

  $\Rightarrow$ unbroken generators T^a with $\eta^a = 1$ and broken generators $T^{\hat{a}}$ with $\eta^{\hat{a}} = -1$
- Kaluza–Klein decomposition: zero modes of $A_5^{\hat{a}}$ are the Higgs fields and A_μ^a are the gauge bosons of the 4d low–energy effective theory at the orbifold boundaries
- Example: $SU(3) \longrightarrow SU(2)\times U(1)$ with $V = (0,\sqrt{3})$
  The gauge bosons branch as $\mathbf{8} = \mathbf{3}_0 + \mathbf{2}_1 + \mathbf{2}_{-1} + \mathbf{1}_0$. There are two Higgs fields in the *fundamental* representation of the unbroken $SU(2)$ gauge group with $U(1)$ charges ± 1.

The boundary terms

- It is plausible that the non–renormalizable 5d orbifold gauge theory defined on the strip I_0 and regularized on a lattice can be described by a continuum Symanzik effective action in some range of lattice spacings (scaling region)
- Symanzik studied the renormalization of the Schrödinger Functional – formulation of quantum field theory with prescribed boundary values [Symanzik, 1981; Lüscher, 1985]
- In the Symanzik effective action renormalization *requires* the inclusion of boundary counterterms

$$\delta S_{\mathrm{b}}[A] \quad = \quad \int \mathrm{d}^4x \sum_i Z_i \left\{ \mathcal{O}_i(x)|_{x_5=0} + \mathcal{O}_i(x)|_{x_5=\pi R} \right\}$$

  where $\dim(\mathcal{O}_i) \leq 4$, since the renormalization factors Z_i go like negative powers of the lattice spacing a

- We can classify the boundary terms in the gauge invariant theory on I_ϵ. They are generated by terms containing $\mathcal{G}$. Then we replace $\mathcal{G}$ by the constant value g

The boundary terms

The Higgs mass counterterm

- The boundary term

$$\mathrm{tr}\{[A_M, g][A_M, g]\} \quad = \quad 2g_0^2 A_5^{\hat{a}} A_5^{\hat{a}}$$

is invariant under gauge transformations of the unbroken subgroup $\mathcal{H}$. If present it generates a power divergent contribution $Z \propto \Lambda^2(g_0^2/R)$ to the Higgs mass

- Explicit 1–loop calculations and a symmetry argument indicate that it is not present [von Gersdorff, Irges and Quiros, 2002,2003]
- This term could be generated from

$$\mathrm{tr}\{D_M\mathcal{G}\, D_M\mathcal{G}\} \quad = \quad 0!$$

- In fact one can easily prove that there are no boundary terms of dimension less than or equal to four
- The lowest dimensional boundary terms are the dimension 5 terms

$$\frac{1}{g_0^2}\mathrm{tr}\{g\, F_{MN}\, F_{MN}\} \quad \text{and} \quad \frac{1}{g_0^2}\mathrm{tr}\{g\, F_{MN}\, g\, F_{MN}\}$$

which appear multiplied by the lattice spacing in the Symanzik action

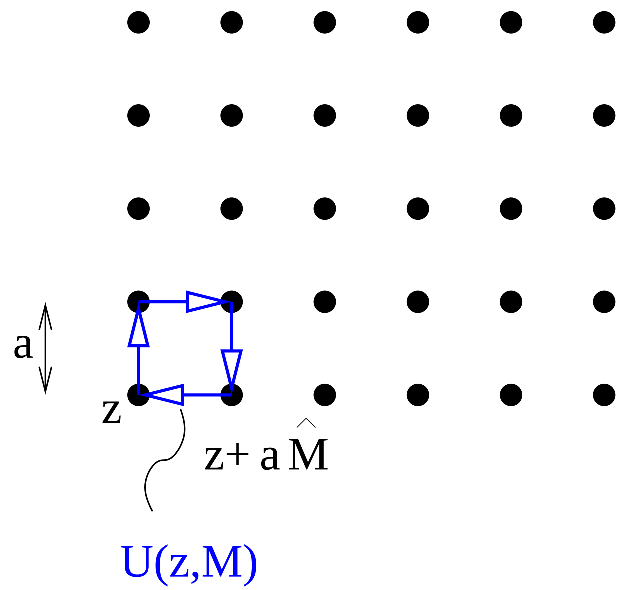
Lattice formulation

Euclidean 5d periodic hypercubic lattice

$$z = a(n_0, n_1, n_2, n_3, n_5)$$

Gauge links $U(z, M) \in \mathrm{SU}(N)$ are the parallel transporter from $z + a\hat{M}$ to z. Under gauge transformations

$$U(z, M) \xrightarrow{\Omega} \Omega(z)\, U(z, M)\, \Omega^\dagger(z + a\hat{M})$$



Wilson plaquette gauge action

$$S_\mathrm{W}[U] = \frac{\beta}{2N} \sum_p \mathrm{tr}\{1 - U(p)\}\,, \qquad \beta = \frac{2N}{g_0^2} a$$

Defining a lattice gauge field through $U(z, M) = \exp\{aA_M(z)\}$ the Yang–Mills action is recovered in the classical continuum limit

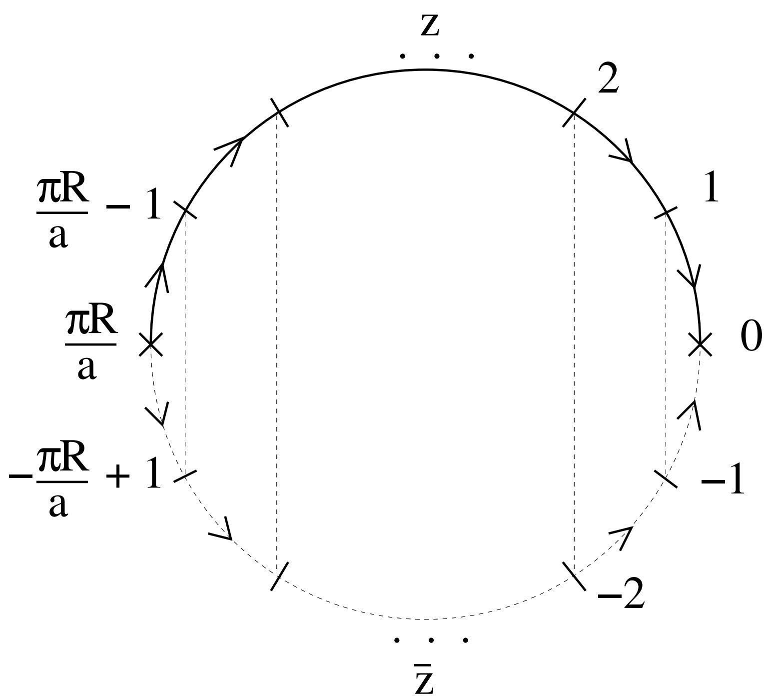
Lattice formulation

Orbifold: $\mathbb{Z}_2$ symmetries

$$\begin{aligned}\mathcal{R}\,U(z,\mu) &= U(\bar z,\mu)\\ \mathcal{R}\,U(z,5) &= U^\dagger(\bar z - a\hat 5,5)\\ \mathcal{T}_g\,U(z,M) &= g\,U(z,M)\,g\end{aligned}$$

Projection, $\Gamma = \mathcal{R}\,\mathcal{T}_g$

$$\frac{1-\Gamma}{2}U(z,M) = 0$$



The lattice orbifold theory can be defined on the strip $I_0 = \{n_\mu, 0 \le n_5 \le N_5 = \frac{\pi R}{a}\}$

$$S_W^{\mathrm{orb}}[U] = \frac{\beta}{2N}\sum_p w(p)\,\mathrm{tr}\{1-U(p)\}\,, \quad w(p) = \begin{cases}\frac{1}{2} & p \text{ in the boundary}\\ 1 & \text{in all other cases.}\end{cases}$$

Lattice formulation

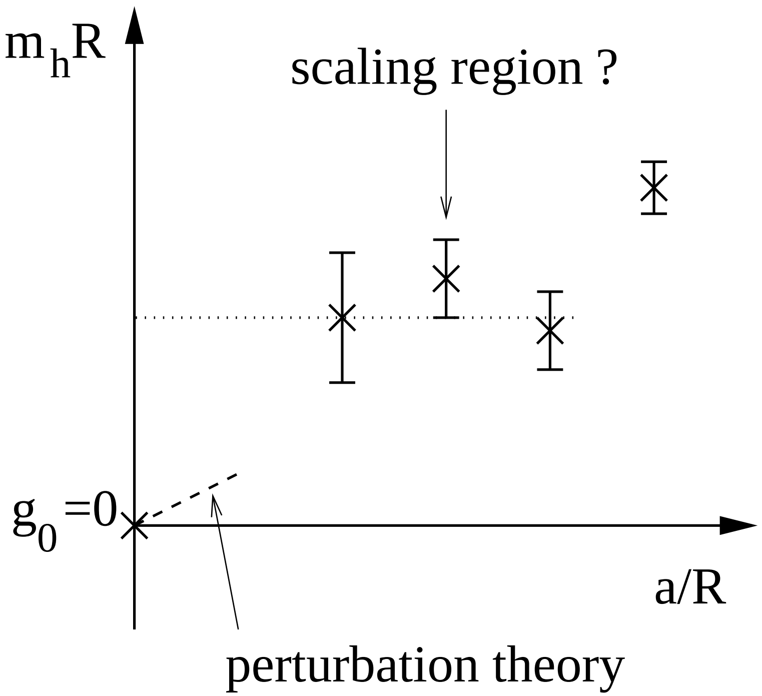
- Dirichlet boundary conditions on the lattice

$$[U(z,\mu), g] \quad = \quad 0 \qquad \text{at } n_5 = 0 \text{ and } n_5 = N_5 = \pi R/a$$

- Dirichlet boundary conditions for the gauge field are recovered in the classical continuum limit
- Information about boudaries is carried by the propagator. The lattice orbifold propagator carries in the continuum limit the Neumann boundary conditions for the gauge field
- The Higgs mass on the lattice can be measured from the exponential fall–off of correlation functions of gauge invariant operators which are odd under the reflection $\mathcal{R}$ [Arnold and Yaffe, 1995]. Compare to the 1–loop formula

$$(m_h R)^2 \quad = \quad \frac{3}{32\pi^4}\frac{g_0^2}{\pi R}\zeta(3)3C_2(G)$$

Lattice formulation



- Higgs mass from perturbation theory depends on the cutoff and is too small for phenomenological application
- Can the theory make sense beyond perturbation theory? does a scaling region exist

$$m_h R \quad = \quad \text{const} + \mathcal{O}(a^2) \quad ?$$

Chiral fermions on the lattice

Nielsen-Ninomiya theorem [H.B. Nielsen and M. Ninomiya, 1981; D. Friedan, 1982]

Free Dirac operator on a Euclidean lattice with lattice spacing a, translation invariant

$$D\mathrm{e}^{ipx}u \quad = \quad \tilde{D}(p)\mathrm{e}^{ipx}u\,, \qquad u \text{ constant Dirac spinor}$$

$\tilde{D}(p)$ is complex 4×4 matrix. The following properties *cannot* hold simultaneously

(a) $\tilde{D}(p)$ is an analytic periodic function of p_μ, period $2\pi/a$
(b) for $p \ll \pi/a$, $\tilde{D}(p) = i\gamma_\mu p_\mu + O(ap^2)$
(c) $\tilde{D}(p)$ is invertible for all non-zero p (mod $2\pi/a$)
(d) D anti-commutes with γ_5

Meaning

(a) locality
(b) correct continuum limit
(c) no additional fermion species (doubling)
(d) continuous chiral symmetry

Chiral fermions on the lattice

Wilson fermions [K.G. Wilson, 1975]

$$D_{\rm w} \quad = \quad \frac{1}{2}\{\gamma_\mu(\partial_\mu + \partial^\star_\mu) - a\partial^\star_\mu\partial_\mu\}$$

Forward and backward difference operators

$$\begin{aligned} \partial_\mu\psi(x) &= \{\psi(x+a\hat{\mu}) - \psi(x)\}/a \\ \partial^\star_\mu\psi(x) &= \{\psi(x) - \psi(x-a\hat{\mu})\}/a \end{aligned}$$

Hermitean γ–matrices $\gamma^\dagger_\mu = \gamma_\mu$, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$

$$\gamma_5 \;=\; \gamma_0\gamma_1\gamma_2\gamma_3 \;=\; \gamma_5^\dagger\,, \quad \{\gamma_\mu, \gamma_5\} \;=\; 0$$

$$\tilde{D}(p) \;=\; i\gamma_\mu \mathring{p}_\mu + \frac{1}{2}a\hat{p}^2\,, \quad \mathring{p}_\mu \;=\; \frac{1}{a}\sin(ap_\mu)\,, \quad \hat{p}_\mu \;=\; \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right)$$

(a),(b) and (c) $\surd$

$\{D_{\rm w}, \gamma_5\} \;\neq\; 0$ $\Rightarrow$ additive mass renormalization, operators of different chiral representations mix

Chiral fermions on the lattice

We follow the Erice lectures by Lüscher `hep-th/0102028`

Chiral projectors $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$, want only one left–handed fermion

$$P_-\psi(x) \;=\; \psi(x)\,, \qquad \overline{\psi}(x)P_+ \;=\; \overline{\psi}(x)$$

Euclidean action $S = a^4 \sum_x \overline{\psi}(x) D_{\mathrm{w}} \psi(x) \longrightarrow a^4 \sum_x \overline{\psi}(x) P_+ D_{\mathrm{w}} P_- \psi(x)$

$$P_+ D_{\mathrm{w}} P_- \;=\; P_+ \frac{1}{2}\gamma_\mu(\partial_\mu + \partial^\star_\mu)\,, \qquad \tilde{D}(p) \;=\; i\gamma_\mu \mathring{p}_\mu$$

(a),(b) and (d) $\surd$

$\mathring{p}_\mu = 0$ for $p_\mu \in \{0, \pi/a\} \Rightarrow$ 16 left–handed fermions!

There is a solution starting from five dimensions . . .

Chiral fermions on the lattice

Dirac operator in 4+1 Euclidean dimensions, s extra coordinate, $\phi(s)$ defines a domain wall of height M and width $1/M$

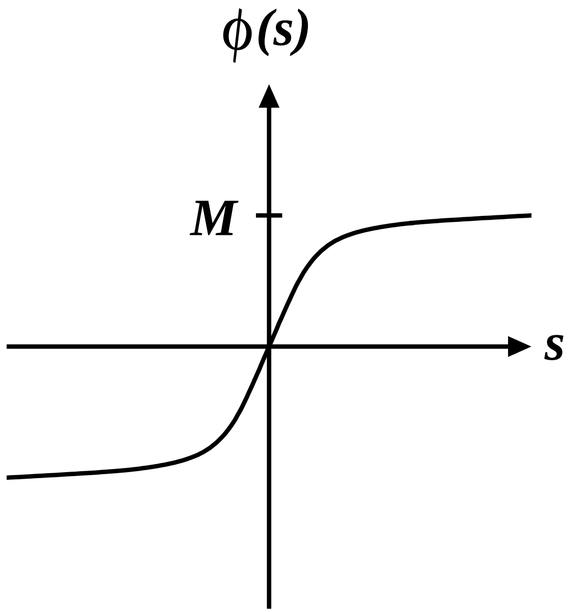
$$D_5 \quad = \quad \gamma_\mu \partial_\mu + \gamma_5 \partial_s - \phi(s)$$

Fermion fields

$$\chi(x,s) \quad = \quad \mathrm{e}^{ipx} u(s)\,, \quad p_\mu \; = \; (iE, \vec{p})$$

Dirac equation $D_5 \chi(x,s) = 0 \longrightarrow$ tower of fermions with masses $m^2 = E^2 - \vec{p}^{\,2}$

- one massless left–handed fermion bound to the domain wall
- fermions (of both chirality) with masses m of order M or larger



dimensional reduction for energies $E \ll M$

Chiral fermions on the lattice

- The domain wall can be replaced by Dirichlet boundary conditions

$$P_+\chi(x,s)|_{s=0} = 0 \quad \text{and} \quad s \geq 0$$

  and $D_5 = \gamma_\mu\partial_\mu + \gamma_5\partial_s - M$. This can be achieved by an orbifold projection
- The construction works on the lattice $M = 1/a$: 5d Wilson Dirac operator $\longrightarrow$ *one* massless left–handed fermion in 4d [Kaplan, 1992]
- What about Nielsen–Ninomiya?
  - On a finite periodic lattice there is a massless right–handed fermion bound to an anti–domain wall. Chiral pairs are exponentially separated in the fifth dimension.
  - The propagator for the left–handed fermion along the boundary $s = 0$ derives from a modified Dirac operator D

$$\{D, \gamma_5\} = aD\gamma_5 D$$

    This is the Ginsparg–Wilson relation [Ginsparg and Wilson, 1982; P. Hasenfratz, V. Laliena and F. Niedermayer, 1998; Lüscher 1998]. Lüscher found a modified chiral symmetry which in the continuum limit $a \longrightarrow 0$ reduces to the usual form.

Conclusion

- We have constructed a 5d orbifold gauge theory from a gauge invariant theory with a spurion field defined in neighborhoods of the orbifold fixpoints
- The breaking of the gauge symmetry at the fixpoints is achieved by shrinking the neighborhoods to points and setting the spurion field to a constant matrix
- We proved that there are no counterterms of dimension less than five localized at the fixpoints of the orbifold
- In particular the Higgs field, identified with some of the 5d components of the gauge field, is protected from power divergent contribution to its mass
- The orbifold theory can be constructed straightforwardly on the lattice. Only Dirichlet boundary conditions on the 4d gauge links at the orbifold fixpoints are needed. The propagator carries the correct Neumann boundary conditions in the continuum limit.

Outlook

- Simulations of the 5d orbifold theory on the lattice
  - Determination of Higgs mass ($\longleftrightarrow$ Debye mass in nonzero temperature field theory), is there a scaling region?
  - Dimensional reduction to a 4d effective theory with weak but appreciable interactions: localization of fields at the orbifold fixpoints? Hosotani mechanism for electroweak symmetry breaking?
- Fermions
  - Orbifold formulation of fermions on the lattice inspired by Domain wall
  - Kaplan, 1992: coupling Domain wall fermions to 5d gauge fields is not appropriate since: unwanted 5d gauge field components, propagation of gauge bosons in the bulk
  - Can the orbifold accommodate 5d gauge and fermion fields together?